Quadriken

Für jedes $t \in \mathbb{R}$ sei die Quadrik Q_t definiert durch:

$$Q_t : \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, 4x_1^2 + 4x_2^2 - 4x_1x_2 - 12x_1 + t = 0 \right\}$$

$$x^T A x + 2a^T x + c = 0$$

$$4x_1^2 + 4x_2^2 - 4x_1x_2 \Rightarrow A = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$$

$$-12x_1 \Rightarrow \begin{pmatrix} -6 \\ 0 \end{pmatrix} = a$$

$$c = t$$

$$\chi_A(\lambda) = \det(A - \lambda E_2) = \begin{vmatrix} 4-\lambda & -2 \\ -2 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 - 4$$

$$\chi_A(\lambda) \overset{!}{=} 0 \Rightarrow (4-\lambda)^2 = 2^2 \Rightarrow 4 - \lambda = \pm 2$$

$$\Rightarrow \begin{cases} \lambda_2 = 4 + 2 = 6 \\ \lambda_1 = 4 - 2 = 2 \end{cases}$$

$$\lambda_2 = 6 \Rightarrow (A - 6E_2) \cdot x = 0 \Rightarrow \left[\begin{array}{cc|c} 4-6 & -2 & 0 \\ -2 & 4-6 & 0 \end{array}\right] \Rightarrow$$

$$\Rightarrow \left[\begin{array}{cc|c} -2 & -2 & 0 \\ -2 & -2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$Z_1 : x_1 + x_2 = 0 \Rightarrow x_1 = -x_2 \Rightarrow v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda_1 = 2 \Rightarrow (A - 2E_2) \cdot x = 0 \Rightarrow \left[\begin{array}{cc|c} 4-2 & -2 & 0 \\ -2 & 4-2 & 0 \end{array}\right] \Rightarrow$$

$$\Rightarrow \left[\begin{array}{cc|c} 2 & -2 & 0 \\ -2 & 2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 2 & -2 & 0 \\ \cancel{4}2 & -2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$Z_1: x_1 - x_2 = 0 \Rightarrow x_1 = x_2 \Rightarrow v_1 = \begin{pmatrix}1\\1\end{pmatrix}$

$\hat{v}_1 = \frac{v_1}{|v_1|} = \frac{1}{\sqrt{1^2+1^2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$\hat{v}_2 = \frac{v_2}{|v_2|} = \frac{1}{\sqrt{1^2+1^2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

$\mathbb{F} = (0, \hat{v}_1, \hat{v}_2)$

$\Leftrightarrow \mathbb{F} = \left(0, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right)$

und $F = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} = {}_E K_F$

$\Rightarrow 0 = y^T(F^TAF)y + 2(F^Ta)^T y + \lambda$

$= (y_1, y_2)\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}4 & -2\\-2 & 4\end{pmatrix}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\right)\begin{pmatrix}y_1\\y_2\end{pmatrix} +$

$+ 2\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}-6\\0\end{pmatrix}\right)^T (y_1, y_2) + \lambda$

$= (y_1, y_2)\left(\frac{1}{2}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}2 & 6\\2 & -6\end{pmatrix}\right)\begin{pmatrix}y_1\\y_2\end{pmatrix} + 2\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}-6\\-6\end{pmatrix}^T \cdot$

$\cdot (y_1, y_2) + \lambda$

$= (y_1, y_2)\left(\frac{1}{2}\begin{pmatrix}4 & 0\\0 & 12\end{pmatrix}\right)\begin{pmatrix}y_1\\y_2\end{pmatrix} + \frac{1}{\sqrt{2}}(-12, -12)\, y + \lambda$

$= (y_1, y_2)\begin{pmatrix}2 & 0\\0 & 6\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix} + (-6\sqrt{2}, -6\sqrt{2})\begin{pmatrix}y_1\\y_2\end{pmatrix} + \lambda$

$= 2y_1^2 + 6y_2^2 - 6\sqrt{2}\,y_1 - 6\sqrt{2}\,y_2 + \lambda$

$$0 = 2y_1^2 - 6\sqrt{2}\,y_1 - 6y_2^2 - 6\sqrt{2}\,y_2 + \lambda$$

$$= 2\left(y_1 - \frac{3\sqrt{2}}{2}\right)^2 - 9 + \left(6y_2 - 3\sqrt{2}\right)^2 - 3 + \lambda$$

$$= 2\left(y_1 - \frac{3\sqrt{2}}{2}\right)^2 + 6\left(y_2 - \frac{\sqrt{2}}{2}\right)^2 + \lambda - 12$$

sei $z_1 = y_1 - \frac{3\sqrt{2}}{2}$

$z_2 = y_2 - \frac{\sqrt{2}}{2}$

$$\Rightarrow 2z_1^2 + 6z_2^2 + \lambda - 12 = 0$$

Fall 1: $\lambda = 12$

$\Rightarrow$ Euklidische Normalform: $2z_1^2 + 6z_2^2 = 0$

Fall 2: $\lambda \neq 12$, $\lambda < 12$

$$\Rightarrow 2z_1^2 + 6z_2^2 + (\lambda - 12) = 0 \quad \Big| \cdot \frac{1}{\lambda - 12}$$

$$\Rightarrow \frac{2}{\lambda-12} \cdot z_1^2 + \frac{6}{\lambda-12} \cdot z_2^2 + 1 = 0$$