Gram-Schmidt

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1) $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

sei $u_1 = v_1$

$\Rightarrow |u_1| = \sqrt{1^2 + 0^2} = \sqrt{1} = 1$

$\Rightarrow w_1 = \frac{u_1}{|u_1|} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\langle v_2 | w_1 \rangle = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Big| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle = 1 \cdot 1 + 1 \cdot 0 = 1$

$proj_{w_1}(v_2) = \langle v_2 | w_1 \rangle \cdot w_1 = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$u_2 = v_2 - proj_{w_1}(v_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$|u_2| = \sqrt{0^2 + 1^2} = \sqrt{1} = 1$

$\Rightarrow w_2 = \frac{u_2}{|u_2|} = 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow w_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $w_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

2) $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

sei $u_1 = v_1 \Rightarrow |u_1| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$

$\Rightarrow w_1 = \frac{u_1}{|u_1|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\langle v_2 | w_1 \rangle = \langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Big| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \rangle$

$= \frac{1}{\sqrt{2}} (1 + 0 + 0)$

$= \frac{1}{\sqrt{2}}$

$proj_{w_1}(v_2) = \langle v_2 | w_1 \rangle w_1 = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\Rightarrow u_2 = v_2 - proj_{w_1}(v_2) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\Rightarrow u_2 = \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix} \Rightarrow |u_2| = \sqrt{\frac{1}{4} + \frac{1}{4} + 1}$

$|u_2| = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$

$\Rightarrow w_2 = \frac{u_2}{|u_2|} = \frac{2}{\sqrt{6}} \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$\langle v_3 | w_1 \rangle = \langle \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \Big| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rangle = \frac{1}{\sqrt{2}} (0 + 1 + 0) = \frac{1}{\sqrt{2}}$

$proj_{w_1}(v_3) = \langle v_3 | w_1 \rangle w_1 = \frac{1}{\sqrt{2}} w_1 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\langle v_3 | w_2 \rangle = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{6}} (0 - 1 + 2) = \frac{1}{\sqrt{6}}$

$\Rightarrow proj_{w_2}(v_3) = \langle v_3 | w_2 \rangle w_2$

$= \frac{1}{\sqrt{6}} w_2 = \frac{1}{6} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$u_3 = v_3 - proj_{w_1}(v_3) - proj_{w_2}(v_3)$

$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1/6 \\ -1/6 \\ 1/3 \end{pmatrix}$

$= \frac{2}{3} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$\Rightarrow |u_3| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

hier brauchen wir $\frac{2}{3}$ in $|u_3|$ nicht, da $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ unser Richtungsvektor ist.

$w_3 = \frac{u_3}{|u_3|} \Rightarrow w_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$