Übung 1

Sei der Untervektorraum:

$V := \{ M \in \mathbb{R}^{2\times 2} \mid M = M^T \}$

der symmetrischen 2x2-Matrizen

Es seien zwei Basen von V gegeben durch:

$B: B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$C: C_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, C_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, C_3 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$

weiter sei die lineare Abbildung:

$\alpha: V \to V,\ X \mapsto A^T X + XA\ ,\ A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$

a) Stellen Sie C_2 und $\alpha(B_3)$ als Linearkombination der Basisvektoren aus B dar:

$C_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$= 0 \cdot B_1 + 1 \cdot B_2 + 1 \cdot B_3$

$\alpha(B_3) = ?$

$\alpha(B_3) = \alpha\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = B_3$

b) ${}_B id_C = ?$

${}_B \alpha_B = ?$

${}_B \alpha_C = ?$

$${}_B id_C = \left({}_B C_1, {}_B C_2, {}_B C_3\right)$$

$$C_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \boxed{0} \cdot B_1 + \boxed{0} \cdot B_2 + \boxed{1} \cdot B_3$$

$$\Rightarrow {}_B C_1 = (\boxed{0}, \boxed{0}, \boxed{1})^T$$

$$C_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \boxed{0} \cdot B_1 + \boxed{1} \cdot B_2 + \boxed{1} \cdot B_3 \quad (a)$$

$$\Rightarrow {}_B C_2 = (\boxed{0}, \boxed{1}, \boxed{1})^T$$

$$C_3 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \boxed{2} \cdot B_1 + \boxed{0} \cdot B_2 + \boxed{0} \cdot B_3$$

$$\Rightarrow {}_B C_3 = (2, 0, 0)^T$$

$$\Rightarrow {}_B id_C = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

$\hookrightarrow {}_B C_1$ $\hookrightarrow {}_B C_2$ $\hookrightarrow {}_B C_3$

$${}_B \alpha_B = \left({}_B\alpha(b_1), {}_B\alpha(b_2), {}_B\alpha(b_3)\right)$$

$$\alpha : V \to V,\ X \mapsto A^T X + XA \Leftrightarrow \alpha(X) = A^T X + XA$$

$$\Rightarrow \alpha(b_1) = \alpha\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \alpha(b_1) = \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix} = 4 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \boxed{4} \cdot B_1 + \boxed{1} \cdot B_2 + \boxed{0} \cdot B_3$$

$$\Rightarrow {}_B\alpha(b_1) = (\boxed{4}, \boxed{1}, \boxed{0})^T$$

$$\alpha(b_2) = \alpha\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$$

$$\Rightarrow \alpha(b_2) = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \boxed{1} \cdot B_1 + \boxed{2} \cdot B_2 + \boxed{2} \cdot B_3$$

$$\Rightarrow {}_B\alpha(b_2) = (\boxed{1}, \boxed{2}, \boxed{2})^T$$

$$\alpha(b_3) = \alpha\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \alpha(b_3) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \boxed{0} \cdot B_1 + \boxed{1} \cdot B_2 + \boxed{0} \cdot B_3$$

$$\Rightarrow {}_B\alpha(b_3) = (\boxed{0}, \boxed{1}, \boxed{0})^T$$

$$\begin{array}{ccc} & {}_B\alpha(b_2) & \\ & & \end{array}$$

$$\Rightarrow {}_B\alpha_B = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

$$\begin{array}{ccc} {}_B\alpha(b_1) & & {}_B\alpha(b_3) \end{array}$$

$$\left.\begin{aligned} {}_B\alpha_C &= {}_B id_B \cdot {}_B\alpha_B \cdot {}_B id_C \\ {}_B id_B \cdot {}_B\alpha_B &= {}_B\alpha_B \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow {}_B\alpha_C = {}_B\alpha_B \cdot {}_B id_C$$

$$\Rightarrow {}_B\alpha_C = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4\cdot0+1\cdot0+0\cdot1 & 4\cdot0+1\cdot0+0\cdot1 & 4\cdot2+1\cdot0+0\cdot0 \\ 1\cdot0+2\cdot0+1\cdot1 & 1\cdot0+2\cdot1+1\cdot1 & 1\cdot2+2\cdot0+1\cdot0 \\ 0\cdot0+0\cdot2+0\cdot1 & 0\cdot0+2\cdot1+0\cdot1 & 0\cdot2+2\cdot0+0\cdot0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 8 \\ 1 & 3 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

Übung 2

Sei der Untervektorraum $V := \{ M \in \mathbb{R}^{2\times 2} \mid M = M^T \}$

Seien die Basen:

$B: B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$C: C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, C_2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, C_3 = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$

Gegeben sei die lineare Abbildung:

$\alpha : V \to V,\ X \mapsto A^T X + XA \quad , \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$

a) Stellen Sie C_3 und $\alpha(B_1)$ als Linearkombination aus B dar

$$C_3 = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} = 1 \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= 0 \cdot B_1 + 1 \cdot B_2 + 2 \cdot B_3$$

$$\alpha(B_1) = \alpha\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \alpha(B_1) = 0 \cdot B_1 + 1 \cdot B_2 + 0 \cdot B_3$$

b) ${}_B \mathrm{id}_C = ?$

${}_B \alpha_B = ?$

${}_B \alpha_C = ?$

$_B id_C = ({}_B C_1, {}_B C_2, {}_B C_3)$

$C_1 = \begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0\\0&0\end{pmatrix} + \begin{pmatrix}0&0\\0&1\end{pmatrix} = 1\cdot B_1 + 0\cdot B_2 + 1\, B_3$

$\Rightarrow {}_B C_1 = (1, 0, 1)^T$

$C_2 = \begin{pmatrix}1&-1\\-1&1\end{pmatrix} = \begin{pmatrix}1&0\\0&0\end{pmatrix} + \begin{pmatrix}0&1\\1&0\end{pmatrix} + \begin{pmatrix}0&0\\0&1\end{pmatrix} = 1 B_1 - 1 B_2 + 1 B_3$

$\Rightarrow {}_B C_2 = (1, -1, 1)^T$

aus ⓐ: $C_3 = 0 B_1 + 1 B_2 + 2 B_3$

$\Rightarrow {}_B C_3 = (0, 1, 2)^T$

$\Rightarrow {}_B id_C = \begin{pmatrix}1&1&0\\0&-1&1\\1&1&2\end{pmatrix}$

${}_B \alpha_B = ({}_B \alpha(B_1), {}_B \alpha(B_2), {}_B \alpha(B_3))$

$\alpha(B_1) = \begin{pmatrix}0&1\\1&0\end{pmatrix} = 0 B_1 + 1 B_2 + 0 B_3$ (aus ⓐ)

$\Rightarrow {}_B \alpha(B_1) = (0, 1, 0)^T$

$\alpha(B_2) = \alpha\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&1\\1&2\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} + \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\-1&2\end{pmatrix}$

$\Rightarrow \alpha(B_2) = \begin{pmatrix}-2&2\\2&2\end{pmatrix} = -2 B_1 + 2 B_2 + 2 B_3$

$\Rightarrow {}_B \alpha(B_2) = (-2, 2, 2)^T$

$\alpha(B_3) = \alpha\begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}0&-1\\1&2\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix} + \begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}0&1\\-1&2\end{pmatrix}$

$\Rightarrow \alpha(B_3) = \begin{pmatrix}0&-1\\-4&4\end{pmatrix} = 0 B_1 - 1 B_2 + 4 B_3$

$\Rightarrow {}_B \alpha(B_3) = (0, -1, 4)^T$

$$\Rightarrow {}_B\alpha'_B = \begin{pmatrix} 0 & -2 & 0 \\ 1 & 2 & -1 \\ 0 & 2 & 4 \end{pmatrix}$$

$${}_B\alpha_C = {}_B\mathrm{id}_B \cdot {}_B\alpha_B \cdot {}_B\mathrm{id}_C$$

$$= \begin{pmatrix} 0 & -2 & 0 \\ 1 & 2 & -1 \\ 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2 & -2 \\ 0 & -2 & 0 \\ 4 & 2 & 10 \end{pmatrix}$$